Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department **Qualifying Courses (Mathematics)**		Final Term Exam Date: January 24, 2017 Course: Linear Algebra, EMM 402 Duration: **3** hours
• Answer **All** questions The exam consists of one page		• No. of questions: 5 Total Mark: 60

Question 1

(a)Determine the linearly independent and linearly dependent: 4

(i) u = (1, 2), v = (2, 4) (ii) u = (1, 0, 3), v = (1, 2, 0), w = (2, 2, 1)

(b) If $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ 1 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}$ 8

Find, if possible, A + B, A.B, $A.B^t$, A.C, C.A, |A| and |C|.

Question 2

(a) Show that the eigenvalues of the matrix : $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ are real numbers, 4

where a, b, c are real numbers.

(b)If $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix}$. 8

(i)Show that A is symmetric and find its eigenvalues and eigenvectors.

(ii)Show that its eigenvectors are orthogonal.

Question 3

Write the following expressions in matrix form and determine the type: 12

(a) $P = 4x^2 + 4y^2 + 3z^2 + 2xy - 3xz + yz$

(b) $P = 4xy + 2xz - 2yz - 4x^2 - 3y^2 - 5z^2$

(c) $P = xy - 6xz + 2yz - x^2 + 2y^2 + 2z^2$

Question 4

(a)Write the Hessian matrix of : $f(x, y, z) = e^{2x}\cosh y + x^3 \ln z$. 4

(b) Find $f(A) = \sqrt{A}$ and $g(A) = \frac{1}{\sqrt{A}}$ where $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ 8

Question 5

(a)Write the equations: $a_{11}x + a_{12}y + a_{13}z = b_1$, $a_{21}x + a_{22}y + a_{23}z = b_2$,

$a_{31}x + a_{32}y + a_{33}z = b_3$ in matrix form and discuss the types of solutions. 5

(b)Solve the linear system :

x – 2y + 2z = 1, 2x + y – 3z = 0, 2x – 4y + 4z = 3, 5y – 7z = – 3. 5

(c)Write the matrix of the transformation : L: $R^3 \to R^2$ 2

where L(x, y, z) = (x + 2y, 2x – 3z).

Good Luck *Dr. Mohamed Eid*

Model Answer

Answer of Question 1

(a)(i) au + bv = a(1, 2) + b(2, 4) = (a + 2b, 2a + 4b) = 0

Then a + 2b = 0, 2a + 4b = 0. Then a = – 2b .

Then u and v are linearly dependent.

(ii) au + bv + cw = a(1, 0, 3) + b(1, 2, 0) + c(2, 2, 1)

= (a + b + 2c, 0a + 2b+ 2c, 3a + 0b + c) = 0

Then a + b + 2c = 0, 2b + 2c = 0, 3a + c = 0. Then a = b = c = 0.

Then u, v and w are linearly independent.

--*4-Marks*

(b) $A + B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 2 & 0 \end{bmatrix}$. $A.B,\ A.C,\ |A|$ are not exist.

$A.B` = \begin{bmatrix} -3 & 11 & -5 \\ 0 & 4 & 2 \\ -2 & 8 & -3 \end{bmatrix}, \quad C.A = \begin{bmatrix} -1 & -1 \\ 11 & 6 \\ 3 & 6 \end{bmatrix}, \quad |C| = 12$

--*8-Marks*

Answer of Question 2

(a)Proof

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(b)(i)Since $A` = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 5 \end{bmatrix} = A$. Then, it is symmetric.

The characteristic equation is $|A - \lambda I| = 0$

Then $\begin{vmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda)(5-\lambda) - 4(4-\lambda) = 0$

$(4-\lambda)[(2-\lambda)(5-\lambda) - 4] = (4-\lambda)(\lambda-1)(\lambda-6) = 0$

Then the eigenvalues are 1, 4, 6.

From the linear system of equations:

$$\begin{bmatrix} 2-\lambda & 0 & -2 \\ 0 & 4-\lambda & 0 \\ -2 & 0 & 5-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$$

For $\lambda=1$, we get the system of equations:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0,$$

Then $x + 0y - 2z = 0$, $0x + 3y + 0z = 0$, $-2x + 0y + 4z = 0$

From the second equation $y = 0$.

From the first equation $x = 2z$ = any number, put $z = 1$.

Then the corresponding eigenvector is $X_1=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda=4$, we get the system of equations: $\begin{bmatrix} -2 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0$

Then $-2x + 0y - 2z = 0$, $0x + 0y + 0z = 0$, $-2x + 0y + z = 0$

From the first equation y = any number.

From the first and third equations $x = z = 0$. Putting $y = 1$.

Then the corresponding eigenvector is $X_2=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda=6$, we get the system of equations:

$$\begin{bmatrix} -4 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=0,$$

Then $-4x + 0y - 2z = 0$, $0x - 2y + 0z = 0$, $-2x + 0y - z = 0$

From the second equation $y = 0$.

From the first or third equations $z = -2x$ = any number, putting $x = 1$.

Then the corresponding eigenvector is $X_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

The matrix of eigenvectors is $T = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

(ii)We see that $X_1`X_2 = 0 = X_1`X_3 = X_2`X_3$

Then, the three eigenvectors are orthogonal.

---*8-Marks*

<u>Answer of Question 3</u>

(a) $P = X`AX = [x \quad y \quad z]\begin{bmatrix} 4 & 1 & -3/2 \\ 1 & 4 & \frac{1}{2} \\ -3/2 & \frac{1}{2} & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is positive definite.

(b) $P = X`AX = [x \quad y \quad z]\begin{bmatrix} -4 & 2 & 1 \\ 2 & -3 & -1 \\ 1 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is negative definite.

(C) $P = X`AX = [x \quad y \quad z]\begin{bmatrix} -1 & 1/2 & -3 \\ 1/2 & 2 & 1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is indefinite.

---*12-Marks*

<u>Answer of Question 4</u>

(a) $H = \begin{bmatrix} 4e^{2x}\cosh y + 6x\ln z & 2e^{2x}\sinh y & \frac{3x^2}{z} \\ 2e^{2x}\sinh y & e^{2x}\cosh y & 0 \\ \frac{3x^2}{z} & 0 & -\frac{x^3}{z^2} \end{bmatrix}$

---*4-Marks*

(b) $|A - \lambda I| = \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = 0$

Then, the eigenvalues are: $\lambda_1 = 1, \quad \lambda_2 = 4.$

(i) $P(\lambda) = f(\lambda_1)P_1 + f(\lambda_2)P_2 = 1\frac{\lambda-4}{1-4} + 2\frac{\lambda-1}{4-1} = \frac{1}{3}(\lambda+2)$

$F(A) = \sqrt{A} = \frac{1}{3}(A + 2I) = \frac{1}{3}\begin{bmatrix} 4 & 2 \\ 1 & 5 \end{bmatrix}$

(ii) $P(\lambda) = g(\lambda_1)P_1 + g(\lambda_2)P_2 = 1\frac{\lambda-4}{1-4} + \frac{1}{2}\frac{\lambda-1}{4-1} = \frac{1}{6}(-\lambda+7)$

$$G(A) = \frac{1}{\sqrt{A}} = \frac{1}{6}(-A+7I) = \frac{1}{6}\begin{bmatrix} 5 & -2 \\ -1 & 4 \end{bmatrix}$$

---*8-Marks*

<u>Answer of Question 5</u>

(a)The matrix form AX = B : $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

If $G = [A \ \vdots \ B]$

This system has three types of solutions :

(i)One solution if rank A = rank G = 3.

(ii)Infinite number of solutions if rank A = rank G < 3.

(iii)No solution if rank A ≠ rank G.

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(b) $G = \left[\begin{array}{ccc:c} 1 & -2 & 2 & 1 \\ 2 & 1 & -3 & 0 \\ 2 & -4 & 4 & 3 \\ 0 & 5 & -7 & -3 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & -2 & 2 & 1 \\ 0 & 5 & -7 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

It has no solution because rank A = 2 and rank G = 3.

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(c)The transformation is : $L = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

---*2-Marks*

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